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Spinor Fields in the Theory of Relativity and a Generalization of Heisenberg's Nonlinear Spinor Equation

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Abstract

Some years ago it was shown that the nonlinear term of Heisenberg's spinor equation can be derived by torsion of the Minkowski space (Cartan space). This result is applied in the investigations of this paper. As the Heisenberg equation does not show any connection with recent phenomenological theories in high energy physics, like the parton or quark model, the problems of the metric of space-time are discussed from the aspect of fundamental axioms of topology (Hausdorff space). It will be shown that Feynman's relativistic parton theory can be derived by means of a quantised de Sitter space, where the constant curvature can assume only discrete values. It is also possible to derive the Dirac equation from the same mathematical considerations. A nonlinear spinor equation will be formulated which contains the parton theory and the nonlinear term of the Heisenberg equation as different approaches in the theory of elementary particles.

1. *Introduction*

In the present quantum field theory (QFT) there are two profound problems, which have led to the consideration of a smallest length l in theoretical physics:

1. The divergencies of the self-interaction terms for the calculation of the rest mass of the elementary particles. According to relativity theory there is an equivalence between matter, energy and field. Therefore the mass of a particle, which is a measurable magnitude, should be able to be calculated by means of the self-interaction with its own field.
2. The necessity of a discrete mass spectrum for the description and characterisation of elementary particles.

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Heisenberg (1967) and other authors (Tamm, 1969 and Kirzhnits, 1971) have emphasised that the constants c (velocity of light) and $\hbar$ (Planck's constant) are not sufficient for the introduction of a unified field, where the discrete masses represent eigenvalues of this field, as it is impossible to form an expression with the dimension of a mass with these two constants. If the mass is not considered as a given magnitude, then the introduction of l must be taken into account. In recent times many discussions have been dedicated to the possible candidates for l. According to Heisenberg's reasoning l should be of the order

$$l \approx \hbar/Mc \approx 10^{-13}\ \text{cm} \tag{1.1}$$

where M is a baryon mass. Other values for l have also been proposed, for example,

$$l \approx 8\pi(\hbar G/c^3)^{1/2} \approx 10^{-32}\ \text{cm} \tag{1.2}$$

G is the Newtonian gravitational constant. (See report by Kirzhnits (1971).) In previous investigations (Ulmer) we have observed a connection between l, introduced for the quantisation of a De Sitter space and Feynman's relativistic parton theory, which represents a phenomenological extension of QFT, as for the description of the inner structure of elementary particles harmonic oscillators have been assumed. Although we have derived Feynman's parton theory (or quark theory, if we identify partons with quarks), we intend to regard the order of l as an open question, for this question can only be answered by experimental results.

2. *The Nonlinear Term of Heisenberg's Spinor Equation and the Affine Connection*

For the description of elementary particles Heisenberg *et al.* (1959) proposed the nonlinear spinor equation

$$\gamma^\nu \Psi_{,\nu} \pm l^2 \gamma^\nu \gamma (\Psi \gamma_\nu \gamma \Psi) \Psi = 0 \tag{2.1}$$

In this paper $\Psi_{,\nu}$ means partial derivation. Some motivations for equation (2.1) were induced by the fundamental symmetries of the Lorentz and isospin group, *resp.* the β-decay. In 1957 Lee and Yang supposed, and Wu observed, that neutrinos and antineutrinos violate the assumption of parity invariance. The transmutability of elementary particles requires a nonlinear spinor equation, and Heisenberg characterised this situation with: 'Each elementary particle is moving in the field of the other particles'. Some years ago it was shown by Rodichev (1961), Braunss (1964, 1965), and Schmutzer (1964) that the equation (2.1) is able to be interpreted by torsion of space. Rodichev (1961) and Schmutzer (1964) used the postulates:

1. The components of the affine connection (in the Riemann-Cartan space) are not symmetrical, for they are defined by

$$G^\lambda_{\mu\nu} = \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} + C^\lambda_{\mu\nu} \tag{2.2}$$

Hereby $\{{}^{\lambda}_{\mu\nu}\}$ are the well-known Christoffel symbols $\{{}^{\lambda}_{\mu\nu}\} = \{{}^{\lambda}_{\nu\mu}\}$. The $C^{\lambda}_{\mu\nu}$ are the components of the completely antisymmetric tensor of 3rd rank, representing the torsion of space (Cartan space). Equation (2.2) defines a Riemann-Cartan geometry. For the following we need the case, where $C^{\lambda}_{\mu\nu} \neq 0$ and $\{{}^{\lambda}_{\mu\nu}\}$ vanishes ($\{{}^{\lambda}_{\mu\nu}\} \equiv 0$).

2. The metrical invariant R is obtained by contraction $R = g^{\alpha\beta}R_{\alpha\beta}$; and, because $\{{}^{\lambda}_{\mu\nu}\} \equiv 0$, R is given by

$$R = C^{\lambda\mu\nu}C_{\lambda\mu\nu} \tag{2.3}$$

The field equation (2.1) is obtained by the variation principle

$$\delta \int (L - bR)\, d^4x = 0, \qquad b = \text{const} \tag{2.4}$$

The Lagrangian L must be defined by

$$L = \tfrac{1}{2}[\overline{\Psi}\gamma_\mu(\Psi_{,\mu} \mp \Gamma^{\text{Cartan}}_\mu \Psi) - (\overline{\Psi}_{,\mu} \mp \overline{\Psi}\overline{\Gamma}^{\text{Cartan}}_\mu)\gamma_\mu \Psi] \tag{2.5}$$

The components $\Gamma^{\text{Cartan}}_\mu$ of the affine connection in the spinor formalism (that means covariant derivation of a Cartan space with $\{{}^{\alpha}_{\nu\mu}\} = 0$) are defined as follows

$$\Gamma^{\text{Cartan}}_\mu = -\overline{\Gamma}^{\text{Cartan}}_\mu = \tfrac{1}{4} \cdot \sum_{\alpha,\beta} C_{\mu\alpha\beta}\gamma_\alpha\gamma_\beta$$

Now we perform the variation of equation (2.3) with respect to $C_{\lambda\mu\nu}$, then we obtain ($\lambda \neq \mu \neq \nu \neq \lambda$)

$$C_{\lambda\mu\nu} = (1/8b) . (\overline{\Psi}\gamma_\lambda\gamma_\mu\gamma_\nu\Psi) \tag{2.6a}$$

$$\Gamma^{\text{Cartan}}_\mu = (1/32b) . (\overline{\Psi}\gamma_\lambda\gamma_\mu\gamma_\nu\Psi)\gamma^\lambda\gamma^\nu \tag{2.6b}$$

This result for $\Gamma^{\text{Cartan}}_\mu$ we shall use again later. Varying L with respect to $\overline{\Psi}$ leads to the Heisenberg equation

$$\gamma^\nu\Psi_{,\nu} \pm (1/32b)(\overline{\Psi}\gamma_\lambda\gamma_\mu\gamma_\nu\Psi)\gamma^\lambda\gamma^\mu\gamma^\nu\Psi = 0 \qquad (\lambda \neq \mu \neq \nu \neq \lambda)$$
$$= \gamma^\nu\Psi_{,\nu} \pm l^2\gamma^\nu\gamma(\overline{\Psi}\gamma_\nu\gamma\Psi)\Psi = 0, \qquad l^2 = 3b/16$$

The nonlinear term of equation (2.1) appears as a consequence of a torsion of space, representing a universal spin-spin-contact interaction. The same result was obtained by Braunss (1964, 1965), without using the variation principle. This way of interpreting the nonlinear term of equation (2.1) makes new aspects apparent, for example, the relationship to the β-decay and the Weyl equation

$$\gamma^\nu\Psi_{,\nu} = 0 \tag{2.7}$$

which is represented here by four-component Dirac spinors. The heuristic principle which leads to the nonlinear term can be outlined as follows: the field equations for the description of the physical events are partitioned in two *categories:*

1. There exists always an 'intrinsic' system $0'$, in which the field equations are as simple as possible, i.e. linear.
2. This system is related to the 'physical' system (in which the field equations are, in general, not linear) by an affine connection, induced by certain transformation groups. Because all measurements must be performed in the system $\Sigma(x)$, we call it in this paper *observation space.*

Let us now restrict ourselves to local Lorentz transformations, varying from point to point. Since the Lorentz group contains the group of three-dimensional rotations, we can interpret this as a torsion of space. The state-vectors Ψ must be transformed by a continuous transformation, given by

$$\Psi = S\Psi', \qquad L_\nu{}^\mu \gamma_\mu = S^{-1}\gamma_\lambda S \tag{2.8}$$

($L_\nu{}^\mu$ is the general Lorentz transformation). With help of (2.8) the Heisenberg equation becomes

$$\gamma^\lambda S\Psi'_{,\lambda} + (\gamma^\lambda S_{,\lambda}S^{-1} \pm l^2\gamma^\lambda\gamma(\Psi\gamma_\lambda\gamma\Psi))S\Psi' = 0 \tag{2.9}$$

It was shown by Braunss that the expression

$$\gamma^\lambda S_{,\lambda}S^{-1} \pm l^2\gamma^\lambda\gamma(\bar{\Psi}\gamma_\lambda\gamma\Psi) \tag{2.10}$$

must vanish to satisfy the conservation laws (invariance under the Pauli–Gürsey transformation):

$$(\bar{\Psi}\gamma^\lambda\gamma\Psi)_{,\lambda} = 0$$

(conservation of the baryonic charge), and

$$(\bar{\Psi}\gamma^\lambda\Psi)_{,\lambda} = 0$$

(conservation of the electric charge). From equations (2.9) and (2.10) follows the condition

$$\gamma^\lambda S\Psi'_{,\lambda} = 0 \tag{2.11}$$

Multiplying this result on the left by S^{-1}, we obtain the Weyl equation

$$S^{-1}\gamma_\lambda S = \gamma_\mu L_\lambda{}^\mu = \gamma_\mu \frac{\partial x^\lambda}{\partial x^{\mu'}}$$

$$\gamma^\mu\Psi'_{,\mu} = 0$$

This result is significant, as it explains the nonlinear term as a consequence of torsion as well as the relationship between the Heisenberg equation and the Weyl equation for neutrinos and antineutrinos. We shall make use of the covariant spinor formalism, containing torsion of space, for a generalisation of the Heisenberg equation. As we have already mentioned a connection between Feynman's parton theory and a smallest length l, we intend to consider the Heisenberg equation from new aspects. Therefore it is necessary to give some background information about recent phenomenological theories in physics of elementary particles.

3. *Feynman's Relativistic Parton (or Quark) Theory*

In high energy physics much progress was achieved by means of group theory for the classification and characterisation of the observed particles. There have been many attempts to make phenomenological extensions of the usual relativistic (and even nonrelativistic) quantum theory for the description of the internal structure of the particles, which the local QFT describes as 'points'. The harmonic oscillator plays a significant role in the approximation of the inner interaction forces. This kind of approximation is assumed in the parton and quark model and in the dual model. Feynman (1971) introduced for the internal structure of a particle the 'partons', but a successful interpretation by Bjorken (1969) identifies 'partons' with 'quarks'. The Heisenberg equation did not show a connection to these phenomenological theories, and we shall see this fact in this paper.

Now we intend to describe the foundations of the relativistic parton theory developed by Feynman (1971) and coworkers. We emphasise that the application of the theory for the calculation of current matrix elements is not our problem here; we satisfy ourselves with a reference to the above publication (Feynman *et al.*, 1971), because our main interest lies in baryon dynamics. Feynman investigated three interacting partons (or quarks) under the condition that the kinetic energy E of a quark is very small compared to the rest energy: $E^2 \ll m_0{}^2c^4$. The operator of three interacting quarks is defined:

$$K = 3(\sum_{i=1}^{3} P_i{}^2 + (1/108)\Omega^2 \sum_{i,j=1}^{3} (u_i - u_j)^2) + C' \tag{3.1}$$

C' and Ω are constants chosen suitable for baryons. $p_i{}^2$ is the square of the four-vector of the momentum operator of quark i:

$$p_i{}^2 = p_{it}^2 - p_{ix}^2 - p_{iy}^2 - p_{iz}^2$$

$p_{i\mu}$ can be replaced by $-i\hbar\, \partial/\partial u_{i\mu}$ and $u_{i\mu}$ is the conjugate position. Feynman *et al.* (1971) assumed the propagator for baryons to be K^{-1}. $P = P_1 + P_2 + P_3$ can be separated from the external motion. With these assumptions and the following substitutions

$$P_1 = (1/3)\,.P - (1/3)\xi; \qquad P_2 = (1/3)\,.P + (1/6)\xi - (1/2\,.\,3^{1/2})\eta;$$
$$P_3 = (1/3)\,.P + (1/6)\xi + (1/2\,.\,3^{1/2})\eta; \qquad u_1 = R - 2x;$$
$$u_2 = R + x - 3^{1/2}\,.y; \qquad u_3 = R + x + 3^{1/2}y;$$

equation (3.2) can be diagonalised:

$$K = P^2 - M^2 \tag{3.2}$$

Feynman called M^2 the mass square operator:

$$-M^2 = (1/2).(\xi^2 + \eta^2) + (1/2).\Omega^2(x^2 + y^2) + C' \tag{3.3}$$

The assumed propagator for the calculation of perturbances becomes:

$$K^{-1} = (P^2 - M^2)^{-1} = \sum_i h_i \frac{1}{p^2 - M_i^2} \bar{h}_i \tag{3.4}$$

where (3.4) is written in terms of Gaussian eigenfunctions $h(\xi, \eta)$ of the harmonic oscillator and $\bar{h}$ means the adjoint to h. The matrix elements between small perturbances $-\delta K$ are given by

$$N_{ij} = \langle \bar{h}_i \,|\, \delta K \,|\, h_j \rangle \tag{3.5}$$

Further calculations of current matrix elements are performed by creation and annihilation operators, for example:

$$\left.\begin{aligned} x &= i(\hbar/2\Omega)^{1/2}(b_x^+ + b_x) \\ y &= i(\hbar/2\Omega)^{1/2}(b_y^+ + b_y) \end{aligned}\right\} \tag{3.6}$$

$$[b_\mu, b_{\mu'}^+] = \delta_{\mu\mu'} \tag{3.7}$$

We should point out that Feynman *et al.* (1971) computed all matrix elements with an additional simplification, because only spacelike excited states were used and the time variable was neglected because the time-integration causes additional difficulties in the formalism. Therefore the parameters Ω and C' have been chosen as they are suitable to facilitate the integration of equation (3.4), but this simplification corresponds to a static quark model, where these 'fundamental' particles would have an infinite lifetime. We shall return to this model again from an axiomatic point of view.

4. *Physical Assumptions for a Generalisation of QFT*

In previous investigations (Ulmer) we have been able to show that a generalisation of the usual quantisation procedure $[x, p_x] = i\hbar$ will lead to a quantised De Sitter space where the (constant) curvature and the spectrum of the space-time variables can only assume discrete values. The results of this mathematical model permit a nontrivial comparison with Feynman's theory. Therefore we intend to summarise the arguments and physical assumptions of the mathematical model. The usual relativistic quantum theory describes a particle without any structure, and interactions between the particles are assumed to be 'point-interactions'. The local QFT and the classical theory of relativity require the satisfaction of the separation axioms in topology (Hausdorff space). We wish to analyse the validity of these axioms from the aspect of the measurement procedure in both theories. A spinor field Ψ_α is quantised with a relation, satisfying local commutativity. For each $\epsilon > 0$ there exist spacelike distances $|x - x'| < \epsilon$ and the Fermion field must be quantised by

$$\Psi_\alpha(x)\Psi_\beta^+(x') + \Psi_\beta^+(x')\Psi_\alpha(x) = \delta_{\alpha\beta}\delta(\epsilon) \qquad (4.1)$$

$\delta(\epsilon)$ is the δ-distribution and x' a continuous neighbourhood of each spacepoint x. The request of relativistic invariance is satisfied in the following sense: We assume two different measurement systems Σ and $\tilde{\Sigma}$, which are *classical* systems. Hence we are able to state the position and relative velocity of one measurement apparatus in the reference system of the other apparatus by a Poincaré transformation

$$\tilde{x}_\mu = L_\mu{}^\nu x_\nu + d_\mu \qquad (\mu, \nu = 1, \ldots 4) \qquad (4.2)$$

When the system Σ measures the state-vector Ψ, the system $\tilde{\Sigma}$ will measure $\tilde{\Psi}$, and a unitary transformation U must exist, thus the scalar product in the Hilbert space will remain unchanged in the two frames:

$$\left.\begin{aligned} |\tilde{\Psi}\rangle &= U(L_\mu{}^\nu, d_\mu)\,|\Psi\rangle \\ \langle\tilde{\Psi}\,|\,\tilde{\Psi}\rangle &= \langle\Psi\,|\,\Psi\rangle \end{aligned}\right\} \qquad (4.3)$$

The relation (4.1) is the result of Pauli's conclusion that Fermions must be quantised in accordance with the Pauli principle, otherwise the energy of the quantised field would not remain *positive definite.* Pauli (1941) considered the general case of a situation, given first by the Dirac equation

$$\left.\begin{aligned} \gamma^\nu\Psi_{,\nu} &= (m_0 c/\hbar)\Psi \\ \gamma_\nu\gamma_\lambda + \gamma_\lambda\gamma_\nu &= 2\tilde{g}'_{\nu\lambda} \\ \tilde{g}_{\nu\lambda} &= \epsilon_\nu\delta_{\lambda\nu}; \qquad \epsilon_1 = \epsilon_2 = \epsilon_3' = -\epsilon_4 = 1 \end{aligned}\right\} \qquad (4.4)$$

This equation for electrons is connected with serious difficulties, concerning the stability of those particles if the Pauli principle is neglected. According to relativity theory we may attach the mass m or energy mc^2 to an electromagnetic field by coupling the electron to its own field. An equilibrium between radiation and self-energy of the electron would never be possible, because it would lose its positive definite energy in a very short time ($\approx 10^{-10}$ sec) by emitting electromagnetic waves. The self-interaction leads to divergencies, although the equivalence between matter and radiation field would associate this term with a finite value mc^2. From this equivalence between matter and every kind of energy in relativity follows the conclusion that the Pauli principle is equivalent to the requirement of the stability of matter, and Dirac solved the above problem with the hole theory, in which the Pauli principle must be assumed. However, the lack of stability of matter is not only a problem of QFT, where divergencies appear to be particularly important, but nonrelative quantum theory and electrodynamics also lead to divergencies. Because of the constant number of particles the above problems do not cause profound difficulties.

From the aspects of measurement processes relativity theory and nonrelativistic quantum theory have similar starting points, although quite different theories are obtained:

1. In the theory of relativity we consider only *macroscopic* systems, for example, measurement apparatuses. For the definition of any rest system Σ' we need a light beam in order to perform a time-synchronisation. We have to apply an interacting field to determine the space-time coordinates and the relative speed. Only then can we make use of the Lorentz transformation, mapping the space-time continuum onto itself

$$\left.\begin{aligned} x^{\mu\prime} &= L_\nu{}^\mu x^\nu, \ \det|L_\nu{}^\mu| = +1 \\ (x'_\mu)(x^{\mu\prime}) &= x_\mu x^\mu = x^2 + y^2 + z^2 - c^2t^2 \end{aligned}\right\} \tag{4.5}$$

With the help of the definition of a rest system Σ' and relative speed we are able to define the energy-momentum relation

$$p_\nu p^\nu = (p'_\nu)(p'_\nu) \Rightarrow E^2 = p^2c^2 + m_0{}^2c^4 \tag{4.6}$$

Here the rest system Σ' refers to a macroscopic system and the interacting field must remain invariant under the transformation (4.5): For example, the electromagnetic field, described by the Maxwell equations.

2. In quantum theory we also need an interacting field for the definition of the space coordinates of a particle. The time coordinate is assumed to be Galileian where $t = t'$ always holds. A consideration of Heisenberg's uncertainty relation makes a fundamental difficulty apparent. The necessary field, described by the Maxwell equations, which remain invariant under the Lorentz transformation, has to obey the uncertainty postulate: 'Heisenberg's uncertainty relation for a material particle, for example, an electron, can be formulated only when the light beam (or any other interacting field) also satisfies the uncertainty principle. If there were no restriction on Δx and Δp for a *light beam*, we would be able to determine the position of the material particle to an infinite degree of accuracy by a well-localised beam without transferring an appreciable momentum uncertainty in an uncontrollable way.' (Sakurai, 1967.)

This connection between measurement processes in quantum theory, where the space coordinates of a material particle are not a given magnitude, and measurement process in relativity theory, where the time coordinate t (and in particular the 'eigen-time' of a particle) is not a given magnitude, shows the starting point of our physical assumptions. We must formulate a mathematical model, in which neither the space variables x, x' nor the time variables t, t' play an exceptional role. The definition of a rest system $\Sigma(x^{\nu\prime})$ for microparticles and the application of the Lorentz transformation between a rest system of such a particle and the measurement system must be restricted. It is an absolute requirement that experimental results, obtained in different observation systems, must be relativistic invariant with respect to the macroscopic systems.

Because the light beam has to obey the uncertainty principle, the transformation group (4.5) and (4.6) can only contain statistical information in the area of subatomic structures and the problem of time-synchronisation between microscopic particles and the reference frame of a measurement apparatus induces further physical and mathematical consequences. We mention once again the Pauli principle. The connection between measurement process in quantum theory and time-synchronisation in relativity theory permits, in the author's opinion, a qualitative understanding of the Pauli principle. In both cases we need a light beam (or any other interacting field), on which we must put the requirement that the field must already obey the uncertainty principle. For this reason there is no possibility for a time-synchronisation of micro-particles to determine the 'eigen-time' $t_1', t_2' \ldots$ and the time $t_1, t_2 \ldots$, registered by the measurement frame, and information about $\Psi(t_1)$, t_1' and $\Psi(t_2)$, t_2' cannot be obtained in one measurement. This agrees with the exclusion principle, which states that no more than one particle can occupy a state-vector Ψ at any one time. (See previous investigations (Ulmer).)

The use of a rest mass (or rest energy) m_0 is only justified in the rest system $\Sigma'(x^{\nu'})$, in which the determination can be performed only in classical systems. When one considers the interaction procedure for the time-synchronisation for a microparticle, the unrestricted application of equation (4.6) must be regarded as problematic, if the uncertainty principle is included from the beginning. In a nonrelativistic approach the problem of a rest mass does not exist, as the observed mass does not depend on the relative velocity.

These problems concerning the rest frames of microparticles and the observation frames, are also of interest in the heuristic principle for the interpretation of the nonlinear term of the Heisenberg equation by Braunss, where the partition in 'intrinsic' system and observation system led to the nonlinear term, describing specific kinds of interactions. This partition can be classified in two *categories*, as there must be mappings between the different systems, e.g.

$$f: x \to x'$$

With respect to conservation laws in physics those mappings are of interest where certain mathematical structures are preserved. In relativity theory and in local QFT it is assumed that all spaces satisfy the separation axioms (Hausdorff space). The relation (4.1) refers to a continuous neighbourhood, and this condition $|x - x'| < \epsilon$ is also satisfied by the Lorentz group $x'_\alpha x^{\alpha\prime} = x_\alpha x^\alpha$, where $x^{\alpha(\prime)}$ are four-vectors and each of them may assume continuous values. In relativity it is also assumed that the rest frame $\Sigma'(x^{\alpha\prime})$ and any measurement frame can coincide at a special $t = t' = 0$ and $x = x' = 0$. However, we have observed that the definition of a rest system $\Sigma'(x^{\nu\prime})$ of any microparticle and the time-synchronisation cause profound difficulties, if the uncertainty principle is taken into account from the beginning. Because the fundamental transformation group of the theory of relativity should be maintained, but not in an unrestricted way and for a continuous set of x^ν and $x^{\nu\prime}$, the transformation (4.5) and (4.6) must assume the character of an operator equation. We have already stated that the change of one reference system $\Sigma(x^\nu)$ to any other system $\tilde{\Sigma}(\tilde{x}^\nu)$ requires the information about the distance between the two systems and the relative velocity: only then are we able to formulate a one-to-one-valued mapping between the two systems

$$f^{\lambda}: \Sigma(x^{\nu}) \to \tilde{\Sigma}(\tilde{x}^{\lambda})$$
$$f^{-1\lambda}: \tilde{\Sigma}(\tilde{x}^{\nu}) \to \Sigma(x^{\lambda})$$

If we consider mappings between classical systems, then all informations for the determination of the different systems are known and the distances between the systems $\Sigma(x^{\lambda})$ and $\tilde{\Sigma}(\tilde{x}^{\lambda})$ represent a given magnitude. As the metric of a space is given by the definition of the distance between the elements of a set, the metric is also determined. If we consider a classical measurement system $\Sigma(x^{\lambda})$ and a rest system $\Sigma'(x^{\lambda'})$ of a microparticle or the rest systems between any two microparticles as reference frames, where the precise determination of all informations for the mappings f and f^{-1} is impossible due to the uncertainty principle, the metric of space-time represents an eigenvalue problem of an operator equation, and x^{ν} and $x^{\nu'}$ cannot assume, in general, continuous values, for the distances (or the metric) of space-time is not a given magnitude. In this case, the separation axiom is not satisfied, and we can see this immediately, as we can interpret all metrical spaces as topological spaces, if we define the distances by the means of the neighbourhoods and the separation axiom is always satisfied in the case of a metrical space.

The set of all real numbers (or the field of real numbers) shall be noted with $\mathbb{R}$. Now let the four-dimensional manifold

$$V(\mathbb{R})$$

be a vector space on the field of real numbers $\mathbb{R}$, which belongs to the measurement system $\Sigma(x^{\nu})$. As we can define on the same field $\mathbb{R}$ any other vector space

$$\tilde{V}(\mathbb{R})$$

belonging to $\tilde{\Sigma}(\tilde{x}^{\nu})$, which is called isomorph to V (or the both reference systems Σ and $\tilde{\Sigma}$ are equivalent), if the following mappings between the vector spaces on $\mathbb{R}$ exist:

$$\left.\begin{aligned} f&: V(\mathbb{R}) \to \tilde{V}(\mathbb{R}) \\ f^{-1}&: \tilde{V}(\mathbb{R}) \to V(\mathbb{R}) \end{aligned}\right\} \tag{4.7}$$

In special theory of relativity the above mappings are specialised to linear transformations:

$$\left.\begin{aligned} x^{\nu} &= L_{\lambda}{}^{\nu}\tilde{x}^{\lambda}, \qquad & x^{\nu} \in V(\mathbb{R}) \\ \tilde{x}^{\lambda} &= \tilde{L}_{\nu}{}^{\lambda}x^{\nu}, \qquad & \tilde{x}^{\lambda} \in \tilde{V}(\mathbb{R}) \end{aligned}\right\} \tag{4.8}$$

The scalar product of the vector space $V(\mathbb{R})$ is a mapping

$$g: V(\mathbb{R}) \to \mathbb{R} \tag{4.9}$$

In the case of special relativity (4.8) we obtain

$$\left.\begin{array}{c}\tilde{g}_{\mu\nu}x^\nu x^\mu = x^2 + y^2 + z^2 - c^2t^2 = S^2 \\ S^2 \in \mathbb{R}\end{array}\right\} \quad (4.10)$$

$\tilde{g}_{\mu\nu}$ is defined as in equation (4.4) by

$$\tilde{g}_{\mu\nu} = \epsilon_\mu \delta_{\mu\nu}, \qquad \epsilon_1 = \epsilon_2 = \epsilon_2 = -\epsilon_4 = 1 \quad (4.11)$$

and assumes diagonal form. If we abandon the group of linear transformations of vector spaces with four-dimensional manifold, we cannot state a mapping $\tilde{g}\colon V \to \mathbb{R}$ which can always be diagonalised, and we shall use the notation g instead of $\tilde{g}$, e.g. the scalar product in the Riemannian geometry

$$dS^2 = g_{\mu\nu}\, dx^\nu\, dx^\mu \quad (4.12)$$

if an arbitrary transformation $x^{\mu\prime} = x^{\mu\prime}(x^\nu)$ is considered. The metric, induced by (4.9) and (4.10), is called pseudo-Euclidean, because the Euclidean space $\mathbb{R}^4$ requires a mapping into $\mathbb{R}_+$:

$$\left.\begin{array}{lll}\tilde{g}\colon \mathbb{R}^4 \to \mathbb{R}_+, & & \mathbb{R}_+ \subset \mathbb{R}, \\ x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2 > 0 & \text{and} & \mathbb{R}_+ = \{x/0 \leqq x < \infty\}\end{array}\right\} \quad (4.13)$$

In special (and in general) theory of relativity we have to consider continuous mappings of isomorph vector spaces V with four-dimensional manifold on the field of real numbers into the same field $\mathbb{R}$, as stated in equation (4.7): $g\colon V(\mathbb{R}) \to \mathbb{R}$.

The field of real numbers $\mathbb{R}$ plays a very important role in the mathematical theories of metrical spaces, because it forms a complete linear space (Banach space), satisfying the separation axiom (Hausdorff space). We shall observe now that in those cases, where several reference frames cannot be fixed by one measurement, e.g. any observation system Σ and a rest frame Σ', the separation axiom is not satisfied and the metric, induced by the mappings of the vector spaces $V(\mathbb{R})$, $V'(\mathbb{R})$ into $\mathbb{R}$, is not a given magnitude. Therefore we consider any two points x and x', where $x' \in \mathbb{R}$ and $x \in \mathbb{R}$. The set of all possible x' shall be given by mappings of $\mathbb{R}$ onto itself.

$$f\colon \mathbb{R} \to \mathbb{R} \Rightarrow f\colon x \to x'$$

The separation axiom requires that for any two points x, $x' \in \mathbb{R}$ there exist neighbourhoods $N(x) \subset \mathbb{R}$ and $N'(x') \subset \mathbb{R}$, satisfying the condition

$$N'(x') \cap N(x) = \phi \quad (4.14)$$

If these neighbourhoods exist, satisfying the defining axiom of a Hausdorff space, we can conclude further: the coincidence of different reference systems Σ and Σ' at $x = x' = t = t' = 0$ is now possible, as the fixation of any point x or t and of points in any neighbourhood of this point x or t requires the satisfaction of (4.14). The topology of $\mathbb{R}$ is separated and forms a complete linear space, as each Cauchy sequence is convergent against an element of $\mathbb{R}$: for each x_n and $x_m \in \mathbb{R}$, there exists $\epsilon > 0$ so that $|x_n - x_m| < \epsilon$ for $n(\epsilon) > n_0$, $m(\epsilon) > m_0$, where follows $\lim_{n \to \infty} x_n \to x \in \mathbb{R}$ for all sequences $\{x_1, \ldots, x_n\} \subset \mathbb{R}$

The topology of $\mathbb{R}$ is used for the definition of the completeness of normed vector space, e.g. the Hilbert space H. In QFT the validity of the separation axiom is also assumed, because the axioms (4.1)-(4.3) form the basis of local QFT.

There is considered a separated Hilbert space H, formed by the set of state vectors $\{\Psi_1 \ldots \Psi_j \ldots\}$, where $\Psi_j \in H_j$ and each H_j is separated. The direct sum $\bigoplus_{j=0}^{\infty} H_j$ also forms the separated Hilbert space H. The metric is induced by the mapping

$$g: H(\mathbb{R}) \to \mathbb{R}_+$$

or:

$$\sum_{j=0}^{\infty} \| \Psi_j \|_2 \doteq \alpha, \longrightarrow \alpha \in \mathbb{R}_+$$

The topology of $\mathbb{R}_+$ is separated, as $\mathbb{R}_+ \subset \mathbb{R}$, defined in (4.13).

The anticommutator rule (4.1) requires local commutativity, as the δ-distribution vanishes for $\epsilon \neq 0$: $[\Psi^+(x'), \Psi(x)]_\pm = \delta(\epsilon)$, where $\epsilon > 0$ and for all spacelike distances $|x - x'| < \epsilon$ must be satisfied. Streater & Wightman (1964) emphasised in their investigations on the axioms of local QFT that the request of local commutativity is perhaps too strong an assumption. If it is not satisfied, then every commutator does not vanish for all distances of space-time points, not only where $x = x'$:

$$[\Psi^+(x'), \Psi(x)] \neq 0, \qquad |x - x'| > 0 \tag{4.15}$$

Recently some authors (see Kirzhnits (1971)) investigated commutator rules of the form

$$\Psi(x)x' - x'\Psi(x) \neq 0 \tag{4.16}$$

to prevent local commutativity. In view of our investigations on the Pauli principle the introduction of commutator rules of the form

$$\Psi(t)t' - t'\Psi(t) \neq 0 \tag{4.17}$$

or more general:

$$\Psi(x^\nu)x^{\mu\prime} - x^{\mu\prime}\Psi(x^\nu) \neq 0$$

would be of interest, since such relations prohibit information about elementary events at the same time. They also express the impossibility of a time-synchronisation. From an algebraic and axiomatic point of view the relation (4.17) or (4.16) is unsatisfactory, as the algebraic foundation is unclear. The relations (4.16) and (4.17) can be specified in more detail as

$$[\phi_\alpha(x^\nu), \phi_\beta^+(x^{\mu\prime})]_\pm = \delta_{\alpha\beta} K_{\alpha\beta}(x^\nu, x^{\mu\prime}) \tag{4.18}$$

$\Sigma'(x'^{\mu})$ means the rest frame or 'intrinsic' system, $\Sigma(x^{\nu})$ any observation system. In equations (4.16)-(4.18) x'^{μ} and x^{ν} represent operators, which cannot commute with each other. For giving an algebraic foundation, we have discussed in previous investigations (Ulmer) the commutator

$$x^{\nu}\gamma_{\mu}x^{\mu}{}'(x^{\nu}) - \gamma_{\mu}x^{\mu}{}'(x^{\nu})x^{\nu} = \gamma^{\nu}l^{2} \tag{4.19}$$

x^{ν} and x'^{μ} are real and represent operators, where the separation axiom is not satisfied, in general, because a fixation of any space-time point x^{ν} of the vector space $V(\mathbb{R})$ does not permit a precise determination of the distances to all points x'^{μ} of $V'(\mathbb{R})$, which may form neighbourhoods of the space-time point x^{ν}. However, if a microsystem is measured by different classical systems Σ and $\tilde{\Sigma}$, where the relative velocities and positions between the two frames must always be given, as they are classical systems, the above mathematical relation (4.19) must be relativistic invariant in the same sense, mentioned already in our discussions about relativistic invariance of local QFT. Relativistic invariance is obtained when the introduced symbols γ_{μ} and γ^{ν} agree with Dirac matrices (metrical bispin tensors). By means of the commutator (4.19) we can form operator-valued functionals and investigate $K_{\alpha\beta}(x^{\nu}, x^{\mu}{}')$, related to the commutator (4.18). In the context of (4.18) and (4.19) we may mention that Heisenberg (1967) introduced the noncanonical quantisation procedure

$$[\Psi_{\alpha}(x), \Psi_{\beta}^{+}(x')]_{+} = \rho(x, x')\delta_{\alpha\beta}\delta(x - x') \tag{4.20}$$

to avoid divergencies, as the spinor equation (2.1) cannot be renormalised. In this procedure $\rho(x, x')$ must vanish for $x = x'$, because then the δ-distribution becomes infinite. It is interesting to note that the condition $K_{\alpha\beta} = \rho(x, x')\delta_{\alpha\beta}\delta(x - x')$ classifies a special set of ϕ-functionals, where (4.18) is satisfied.

The algebraic relation (4.19) can be transformed into a field equation by means of a relation for differential operators: from $AB - BA = c$ follows that the transformation $A \to \partial/\partial B$ yields

$$c\left[\frac{\partial}{\partial B}B - B\frac{\partial}{\partial B}\right]F(B) = cF(B) \tag{4.21a}$$

If $g(B)$ is an arbitrary function, commuting with B, the above relation is also satisfied with

$$A \to \frac{\partial}{\partial B} \pm g(B), \qquad [B, g(B)] \equiv 0 \tag{4.21b}$$

Applying (4.21a, b) to relation (4.19) we obtain the field equation

$$\gamma_{\mu}\cdot(x^{\mu}{}'(x^{\nu}, p_{\nu}))\phi = -l^{2}\gamma^{\nu}[\phi_{,\nu} \pm \Gamma_{\nu}\phi], \qquad [\Gamma_{\nu}(x^{\lambda}), x^{\lambda}] \equiv 0 \tag{4.22}$$

Γ_{ν} is an arbitrary function, which must be chosen so that (4.22) satisfies covariant derivation in the spinor formalism (affine connection in the spinor formalism). In the most general case x'^{μ} represents the mapping $f^{\mu}: x^{\nu} \to x^{\mu\prime}$ and depends also on the relative velocity (or momentum) to the system $\Sigma(x^{\nu})$:

$$x^{\mu\prime} = x^{\mu\prime}(x^{\nu}, p_{\nu}) \tag{4.23a}$$

The affine connection Γ_{ν} in the spinor formalism has been considered very thoroughly by Schmutzer (1964) and other authors (Rodichev, 1961 and Braunss, 1964, 1965). It can be split up in two parts

$$\Gamma_{\nu} = \Gamma_{\nu}^{\text{Riemann}} + \Gamma_{\nu}^{\text{Cartan}}$$

if space-time has Riemannian geometry (curvature) and torsion (Cartan space), introduced in Section 2, equations (2.2), (2.3) and (2.6). A restriction of (4.23a) to the inhomogeneous Lorentz group

$$x^{\mu\prime} = a^{\mu} + L_{\nu}{}^{\mu}x^{\nu} \tag{4.23b}$$

where $L_{\nu}{}^{\mu}$ depends on three relative speeds v_i $(L_{\nu}{}^{\mu} = L_{\nu}{}^{\mu}(v_i))$, yields $\Gamma_{\nu} \equiv \Gamma_{\nu}^{\text{Cartan}}$, as $\Gamma_{\nu}^{\text{Riemann}}$ and $\{{}_{\nu\mu}^{\alpha}\}$ vanish for (4.23b), inducing a space with constant curvature. Now we obtain the following equation

$$\gamma_{\mu}'(a^{\mu} + L_{\nu}{}^{\mu}x^{\nu})\phi = -l^2\gamma^{\nu}[\phi_{,\nu} \pm \Gamma_{\nu}^{\text{Cartan}}\phi] \tag{4.24}$$

The introduced constant l has the dimension of a length, known from the Heisenberg equation (2.1). For the following we wish to consider Feynman's theory again and neglect Γ_{ν}, too, containing torsion of space and making equation (4.24) nonlinear. Thus we get

$$\gamma_{\mu}(a^{\mu} + L_{\nu}{}^{\mu}x^{\nu})\phi = -l^2\gamma^{\nu}\phi_{,\nu} \tag{4.25}$$

Equation (4.25) shows relativistic form invariance with respect to different measurement frames Σ and $\tilde{\Sigma}$. The two systems are determined by the Lorentz transformation $x^{\nu} = A_{\lambda}{}^{\nu}\tilde{x}^{\lambda}$, as they are classical systems. When Σ registers ϕ, then $\tilde{\Sigma}$ will register $\tilde{\phi}$, and there must exist a unitary transformation U between ϕ and $\tilde{\phi}$ of the form $\tilde{\phi}(Ax) = U(A)\phi(x)$. Because of $L_{\nu}{}^{\mu}x^{\nu} + a^{\mu} = a^{\mu} + L_{\nu}{}^{\mu}A_{\lambda}{}^{\nu}\tilde{x}^{\lambda}$ we obtain $x^{\mu\prime} = a^{\mu} + M_{\lambda}{}^{\mu}\tilde{x}^{\lambda}$ and:

$$\gamma_{\mu}(M_{\lambda}{}^{\mu}x^{\lambda} + a^{\mu})\tilde{\phi} = -l^2U(A)\gamma^{\nu}U^{-1}(A)A_{\nu}{}^{\lambda}\tilde{\phi}(\tilde{x}^{\lambda})_{,\lambda} \tag{4.26}$$

where we have used $M_{\lambda}{}^{\mu} = L_{\nu}{}^{\mu}A_{\lambda}{}^{\nu}$ and $\partial/\partial x^{\nu} = A_{\nu}{}^{\lambda} . \partial/\partial\tilde{x}^{\lambda}$. Equation (4.26) has the same form as equation (4.25), if the following conditions are satisfied:

$$U(A)\gamma^{\nu}U^{-1}(A)A_{\nu}{}^{\lambda} = \gamma^{\lambda} \Rightarrow A_{\nu}{}^{\lambda}\gamma^{\nu} = U^{-1}(A)\gamma^{\lambda}U(A) \tag{4.27}$$

The satisfaction of (4.27) is already required by the Dirac equation (4.4), as stated in equation (4.3), and it is evident that U exists.

To learn more about equation (4.25) we perform a further iteration on both sides of equation (4.25). This must lead to a quadratic form:

$$x^{\mu\prime}\gamma_{\mu}\gamma_{\mu}x^{\mu\prime}\phi = (-l^2)^2\frac{\partial}{\partial x^{\nu}}\gamma^{\nu}\gamma^{\nu}\frac{\partial}{\partial x^{\nu}}\phi$$

Using well-known relations for $L_\nu{}^\mu$:

$$(L^{-1})_\lambda{}^\mu L_\nu{}^\lambda = \delta_\nu{}^\mu, \qquad \gamma_\nu = \tilde{g}_{\nu\mu}\gamma^\mu, \qquad L_{\mu\nu} = \tilde{g}_{\mu\lambda}L_\nu{}^\lambda$$

$$\gamma^\nu\gamma_\mu - \gamma_\mu\gamma^\nu = [\gamma^\nu, \gamma_\mu]$$

we obtain:

$$\gamma^\nu\gamma_\mu\left(\frac{\partial x'^\mu}{\partial x^\nu}\phi + x'^\mu\phi_{,\nu}\right) = -l^2\gamma^\nu\gamma^\nu\phi_{,\nu,\nu}$$

By the use of the above defined relation $[\gamma^\nu, \gamma_\mu]$ and the restriction to the linear transformation $x'^\mu = a^\mu + L_\nu{}^\mu x^\nu$ respectively $\partial x'^\mu/\partial x^\nu = L_\nu{}^\mu$ we obtain:

$$\gamma_\mu a^\mu\gamma_\mu x'^\mu\phi + \gamma_\mu\gamma_\mu L_\nu{}^\mu x'^\mu x^\nu\phi - l^2[\gamma^\nu, \gamma_\mu]x'^\mu\phi_{,\nu}$$
$$- l^2[\gamma^\nu, \gamma_\mu]L_\nu{}^\mu\phi \equiv l^4\phi_{,\nu,\nu} \equiv l^4\Box\phi$$

This expression can be transformed by the means of the algebraic commutator relation(4.19):

$$\gamma_\mu\gamma_\mu(a^\mu . a^\mu + x^\mu . x^\mu)\phi - l^2[\gamma^\nu, \gamma_\mu]x'^\mu\phi_{,\nu}$$
$$- l^2[\gamma^\nu, \gamma_\mu]L_\nu{}^\mu\phi = l^4\Box\phi \qquad (4.28a)$$

In order to make equation (4.28a) more convenient and to regard again Feynman's theory we have requested vanishing of the terms

$$\gamma_\mu\gamma_\mu a^\mu L_\nu{}^\mu x^\nu\phi \equiv 0 \qquad (\forall x^\nu \text{ and } L_\nu{}^\mu)$$

This restriction will lead to the condition (4.30). As it is our aim to compare equation (4.28a) with Feynman's theory, stated in Section 3 of this paper, and to formulate a quantised de Sitter space, we request vanishing of the terms

$$-l^2[\gamma^\nu, \gamma_\mu]L_\nu{}^\mu\phi - l^2[\gamma^\nu, \gamma_\mu]x'^\mu\phi_{,\nu}$$

and obtain:

$$\left.\begin{aligned}\tilde{g}_{\mu\mu}(x^\mu x^\mu + a^\mu a^\mu)\phi = l^4\Box\phi \\ (x^2 + y^2 + z^2 - c^2t^2 - a^2)\phi = l^4\Box\phi\end{aligned}\right\} \qquad (4.28b)$$

The γ^ν-matrices must satisfy the definition (4.4);

$$\gamma_\mu\gamma_\lambda + \gamma_\lambda\gamma_\mu = 2\tilde{g}_{\mu\lambda}$$

However, this is only possible as long as we consider linear mappings $f\colon V(\mathbb{R}) \to V'(\mathbb{R})$ or $x'^\mu = L_\nu{}^\mu x^\nu + a^\mu$, because only then exists $g\colon V(\mathbb{R}) \to \mathbb{R}$, where g is diagonalised:

$$g_{\nu\mu} = \tilde{g}_{\nu\mu}$$

In the case of an arbitrary transformation $x'^{\mu} = x'^{\mu}(x^{\nu})$ the above anticommutator becomes

$$\gamma_\nu \gamma_\mu + \gamma_\mu \gamma_\nu = 2g_{\mu\nu} \qquad S = \int dS = \int [g_{\mu\nu}\, dx^\mu\, dx^\nu]^{1/2}, \qquad S \in \mathbb{R} \quad (4.29)$$

This relation has already been discussed by some authors (see Rodichev (1961), Braunss (1964, 1965), Schmutzer (1964) and Datta (1971) in connection with the Dirac equation in the Riemannian space. Because of the difficulties to obtain solutions we restrict ourselves to quadratic forms. The inhomogeneous term a^μ requires the condition

$$\left.\begin{array}{c} -(a)^2 = a_\mu a^\mu = -(a^4)^2 + (a^3)^2 + (a^2)^2 + (a^1)^2 \qquad \gamma_\mu\gamma_\mu a^\mu = 0 \\ a^\mu \gamma_\mu \gamma_\mu = 0 \Rightarrow a^4 = +(a^1 + a^2 + a^3)\, \forall x^\nu, L_\nu{}^\mu \end{array}\right\} (4.30)$$

A diagonalisation of equation (4.28b), which represents a quadratic form, leads to the condition that a^4 is not independent from the remaining a^i, and the condition (4.30) is not required in the case of Riemannian geometry, assuming the relation (4.29) as defining axiom for the metric. Equation (4.28) represents a quantised de Sitter space, where the constant curvature, induced by the term $(a)^2$, can only assume discrete values. This equation was discussed and solved exactly in a previous publication (Ulmer), using harmonic oscillators. The result was a nontrival connection with Feynman's theory, described above. Since in relativistic theories the total mass or energy appears only in the fourth component of a four-vector, which we have to connect with a^4, the remaining a^i $(i = 1, 3)$ must also be related to any kind of mass and lead to particular conditions by the diagonalisation. Therefore they may be considered as 'quasi-particles' and become interesting in the parton (or quark) model.

In recent theories in high energy physics, harmonic oscillators or linear internal forces play an important role. An axiomatic access to those theories we have obtained under the assumptions: the use of a rest mass (or energy) is only justified in the rest system $\Sigma'(x'^{\mu})$, related to that the self-interaction of an elementary particle can be defined. The uncertainty relation permits the exact knowledge of either the measurement frame $\Sigma(x^\nu)$ or the relative velocity v_i, but not both things by one measurement, required for a strict application of the Lorentz transformation. Therefore we have considered this transformation as a term of an operator equation, representing the self-interaction with respect to the rest frame. The linear mapping $f: x \to x'$ represents an approximation, where $\{^{\alpha}_{\nu\mu}\} = 0$. Neglecting certain terms we are able to compare the model of a quantised de Sitter space with Feynman's theory. Considering a^i as 'quasi-particles' we emphasise that the satisfaction of SU_3-symmetry with $a^1 = a^2 = a^3$ is an additional assumption of a high symmetry, which may not absolutely be justified. We wish to state here some juxtapositions to Feynman's parton (or quark) theory

$$-M^2 \Rightarrow a_\mu a^\mu \qquad \text{(equation (3.3))}$$

equation (4.28) corresponds to equation (3.2):

$$K^2 = p^2 - M^2$$

$$(x_\mu x^\mu - (a)^2)\phi = l^4 \Box \phi$$

equation (4.28) was solved (Ulmer) by creation and annihilation operators

$$x^\mu = l \,.\, 2^{-1/2} \,.\, [b^\mu + b^{\mu +}], \qquad [b^\mu, b^{\nu +}] = \delta^{\mu\nu}$$

A comparison with equation(3.7) shows

$$l = (\hbar/\Omega)^{1/2} \tag{4.31}$$

From (4.31) we can verify: setting in the value for Ω, we would obtain $1 \approx 10^{-14}$ cm. But we must refer to our remarks in Section 3. Feynman used Ω suitable to neglect time-excited states in equation (3.2), which cause many difficulties and this approximation yields a static quark model of baryon quarks, which would have an infinite lifetime. It is easy to verify that this simplification does not suit a relativistic formalism. The general solution of equation (4.28b) requires besides the spin four quantum numbers for $(a)^2$ (see (Ulmer)) and each variable x^ν of space-time produces one quantum number. Considering the very short lifetime of some hadrons and the hadron resonances ($\lesssim 10^{-23}$ sec), it is quite certain that l must be much smaller and we are, at present, unable to specify the upper limit for l more exactly. Therefore there is some reason to assume that l may be of the order of the fundamental gravitation interaction length (equation (1.2)).

5. *Further Results and an Application to Heisenberg's Nonlinear Spinor Equation*

The neglect of torsion (Cartan space) or the affine connection in the spinor formalism ($\Gamma_\nu \equiv 0$) is justified because we can find an exact solution for equation (4.25). A set of permissible ϕ-functions for equation (4.25) is obtained by

$$\left.\begin{array}{c} \phi = A \exp\left\{-\dfrac{1}{2l^2}\gamma^\nu \epsilon_\nu \gamma_\mu L_\nu{}^\mu x^\nu x^\nu - ik_\nu x^\nu\right\} \\ + \text{Complex conjugation} \end{array}\right\} \tag{5.1}$$

In the case of the ground-state A is a constant amplitude of a four-component spinor, and there results the following identities:

$$\left.\begin{array}{c} \gamma_\mu L_\nu{}^\mu x^\nu = \gamma^\nu \gamma^\nu \epsilon_\nu \gamma_\mu L_\nu{}^\mu x^\nu \\ \epsilon_1 = \epsilon_2 = \epsilon_3 = -\epsilon_4 = 1 \\ \gamma_\mu a^\mu = il^2 \gamma^\nu k_\nu \end{array}\right\}$$

The wave-vector k_ν can be replaced by $p_\nu/\hbar$, where p_ν is the relativistic four-momentum. The solution (5.1) consists of a product of two parts: 1. The

Gaussian wave-functions, which descend rapidly at greater distances of space-time. 2. The plane-wave solution, which agrees with the Dirac equation (4.4) for free particles. For larger distances we obtain quite natural the free particle solution. For the physical interpretation it is interesting to compare the above solution (5.1) of equation (4.25) with the Schrödinger equation for a free particle (in one space-coordinate)

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} = i\hbar\frac{\partial}{\partial t}\Psi \tag{5.2}$$

The general solution of this nonrelativistic equation may be written in plane-waves:

$$\Psi(x, t) = \sum_k U_k \exp\{i(kx - (E/\hbar)t)\} + \text{Complex conjugation} \tag{5.3}$$

It is also possible to represent the free particle solution (5.3) in the form of a Gaussian wave-packet and to formulate the uncertainty relation by means of

$$\langle x \rangle = \int |\Psi|^2 x \, dx$$
$$(\Delta x)^2 = \int (x - \langle x \rangle)^2 |\Psi|^2 \, dx$$

Now we obtain

$$\Psi(x, t) = [(\Delta x)^2/2\pi^3]^{1/4} \int \exp\left\{-k^2(\Delta x)^2 - \frac{i\hbar k^2 t}{2m} + ikx\right\} dk$$
$$= (2\pi)^{-1/4}\left(\Delta x + \frac{i\hbar t}{2m\Delta x}\right)^{1/2} \exp\left\{-\frac{x^2}{4(\Delta x)^2 + (2i\hbar t/m)}\right\} \tag{5.4}$$

which represents a wave-packet of which the centre remains at $x = 0$ while the breadth of the packet increases as t departs from zero in both past and future directions. Using the formulation of equation (5.2) by Feynman integrals of the kind $\Psi(x, t) = \int G_0(x, x', t, t_0)\Psi(x', t_0)\, dx'$, the corresponding Greens-function $G_0(x, x', t_0, t)$ is represented by Gaussian eigenfunctions:

$$G_0 = \left[\frac{-im}{2\pi\hbar(t - t_0)}\right]^{1/2} \exp\left\{\frac{im(x - x')^2}{2\hbar(t - t_0)}\right\} \tag{5.5}$$

The most significant difference between the solution (5.1) of equation (4.25) and the free particle solution (5.4) of the Schrödinger equation *resp.* the free particle propagator $G_0(x, t, x', t_0)$ of it can be characterised by the behaviour of the time coordinate t in the Gaussian functions. In the case of nonrelativistic quantum theory, where point charges, described by a probability behaviour in the space-time-continuum, are considered, the uncertainty relation requires an increasing breadth of the wave packet, when the time increases. A narrow wave packet is related to a precise measurement of the space coordinate x. As there exists no difficulty, arising from the consideration of the rest frame $\Sigma'(x'^{\mu})$ in relativity theory, to satisfy the separation axiom, if the space distance $\langle \Delta x \rangle$ is fixed very precisely, it is obvious that the momentum uncertainty requires an increasing breadth of the packet. In our relativistic considerations, where we have avoided a special treatment of the time coordinate $t = x^4/c$, we have used the uncertainty principle for the fixation of the uncertainty of any neighbourhood $\langle x' \rangle$, in which the rest frame of a microparticle is defined, when any space-time distance has been measured by an observation system $\Sigma(x^\nu)$ with the precision $\langle \Delta x \rangle$ and $\langle \Delta t \rangle$. Any space-time point can be fixed without any uncertainty of the neighbourhoods of that point, and the separation axiom is not satisfied, in general. An elementary particle can no longer be described as a point, if the distances of space-time for the observation of the particle in $\Sigma(x^\nu)$ are finite. This fact is expressed in the solution (5.1) of equation (4.25) by the very rapid descension of the Gaussian functions, when the time coordinate t is starting with zero and increases in both directions.

The algebraic relation (4.19) can also be used for an algebraic relation of the Dirac equation (4.4). For this purpose it is necessary to assume that the rest frame $\Sigma'(x'^\mu)$ does not depend on the system $\Sigma(x^\nu)$. Instead of the mapping $f: x \to x'$ with $x'^\mu = L_\nu{}^\mu x^\nu + a^\mu$ we represent x'^μ by a constant four-vector and regard the Poincaré group (4.23b) as a linear approximation of $x'^\mu = x'^\mu(x^\nu)$, which shall be neglected:

$$x'^\mu = a^\mu + 0(L_\nu{}^\mu x^\nu + \ldots +) \tag{5.6}$$

A constant four-vector can always be chosen so that the amount of only one component does not vanish and the remaining components are made zero:

$$x'^\mu = 0^\mu (\mu = 4, a^1 = a^2 = a^3 = 0) \tag{5.7}$$

$p^{\mu\prime}$ is the momentum in the rest frame $\Sigma'(x'^\mu)$. Substituting (5.7) by the rest mass and rest momentum $p^{\mu\prime}$, we have to make use of the constant h to obtain the suitable dimension of a length

$$\left.\begin{array}{l} x^{\mu\prime} = l^2 p^{\mu\prime}/(\hbar) \\ p^{41} = m_0 c, \qquad p^{11} = p^{21} = p^{31} = 0 \end{array}\right\} \tag{5.8}$$

With help of this expression for x'^μ the commutator (4.19) becomes

$$x^\nu \gamma_\mu p^{\mu\prime} - \gamma_\mu p^{\mu\prime} x^\nu = \hbar \gamma^\nu \tag{5.9}$$

As the coordinates of the rest frame do not depend on any reference system in a neighbourhood of $\Sigma'(x'^\mu)$, the separation axiom is always satisfied for the algebraic relation (5.9), and as a consequence of this fact, the structure of space-time is a continuous one. Applying the transformation (4.21a) to the relation (5.9), the four-component spinor equation (4.4), known as the Dirac equation,

$$\left.\gamma_\mu p^{\mu\prime} \Psi = -\hbar \gamma^\nu \Psi_{,\nu} \Rightarrow \pm \frac{m_0 c}{\hbar} \Psi = \gamma^\nu \Psi_{,\nu} \right\}$$

is obtained, which plays a fundamental role in relativistic quantum theory. This result makes it evident that the algebraic relation (4.19) provides an interesting tool with great flexibility, as the Dirac equation is obtained, if the mapping $f: x^\nu \to x'^\mu = constant$ is considered. In the last time many authors considered the covariant Dirac equation in the Riemannian geometry. That means, between the different observation systems Σ and $\tilde{\Sigma}$ there exist mappings. inducing Riemannian geometry or: the systems Σ and $\tilde{\Sigma}$ are accelerated against each other. However, $\Gamma_\nu^{\text{Riemann}}$ has no consequence for the rest mass m_0, as the system $\Sigma'(x'^\mu)$ does not depend on any observation system. We are able to obtain the covariant Dirac equation by applying the transformation (4.21b) to equation (5.9), where Γ_ν is specialised to $\Gamma_\nu = \Gamma_\nu^{\text{Riemann}}$:

$$\frac{m_0 c}{\hbar}\Psi = \gamma^\nu[\Psi_{,\nu} \pm \Gamma_\nu^{\text{Riemann}}\Psi] \tag{5.10}$$

The covariant Dirac equation (5.10) has been discussed by many authors; for a detailed treatment of (5.10) and the covariant derivation of four-component spinors in the Riemannian space the publication of Schmutzer (1964) may be consulted, in which the symbol $\Gamma_\nu^{\text{Riemann}}$ and its connection with $\{{}^{\alpha}_{\nu\mu}\}$ is investigated, as our main interest lies in the discussion of a generalisation of the Dirac equation (4.4) by means of the torsion (Cartan space). For this purpose we make use of the transformation (4.21b) and specialise Γ_ν to $\Gamma_\nu = \Gamma_\nu^{\text{Cartan}}$. $\Gamma_\nu^{\text{Cartan}}$ represents the most general case of the group of local transformations between the system $\Sigma'(x'^\mu)$ and the observation space $\Sigma(x^\nu)$, in which all measurements must be performed, as it is impossible to measure in the rest frame $\Sigma'(x'^\mu)$ of an elementary particle. The generalised form of the Dirac equation now becomes:

$$\frac{m_0 c}{\hbar}\Psi = \gamma^\nu = \gamma^\nu[\Psi_{,\nu} \pm \Gamma_\nu^{\text{Cartan}}\Psi], \qquad [\Gamma_\nu^{\text{Cartan}}(x^\lambda), x^\lambda] \equiv 0 \tag{5.11}$$

The group of local transformations in equation (5.11), represented by the symbol; $\Gamma_\nu^{\text{Cartan}}$, is based on the vector space $V(\mathbb{R})$ with pseudo-Euclidean metric. The set of all Lorentz transformations, varying from point to point in the Minkowski space, can therefore be understood as a torsion of the Minkowski space. A twisting (or torsion) of space-time (in this restriction a Minkowski space) leads to the concept of spin, represented by a four-component spinor space (Cartain space). In Section 2, we have already given a reference to some publications which appeared some years ago, concerning the torsion of space-time (Cartan space) and the affine connection in the spinor formalism $\Gamma_\nu^{\text{Cartan}}$, of which we have made use for obtaining the covariant derivation with respect to the spinor space. It was our motiviation for giving these reports in Section 2 that the nonlinear term of the Heisenberg equation can be derived by torsion of space-time, which is equivalent to a Lorentz group, varying from point to point and describing a universal spin-spin-contact interaction. In the publication of Braunss (1964, 1965) the relation of the nonlinear Heisenberg equation to the Weyl equation for neutrinos and antineutrinos has been made more distinct:

$$\gamma^\nu \phi_{,\nu} = 0$$

We obtain the Weyl equation by setting

$$x'^{\mu} \equiv 0, \qquad \Gamma_\nu \equiv 0$$

The physical contents of this condition is that the rest mass m_0 will vanish in a system, where the relative speed is zero. This equation has become actual, as the neutrinos and antineutrinos do not show parity invariance, and this behaviour plays an important role in the Heisenberg equation, discussed in Section 2. According to Braunss the Weyl equation refers to the 'intrinsic' system $\Sigma'(x'^{\mu})$ and the nonlinear term of equation (2.1) is induced by torsion. That means, the term $\Gamma_\nu^{\text{Cartan}}$ is responsible for the Heisenberg term, as shown in Section 2.

The partition in an 'intrinsic' system or rest system and a measurement frame (or observation space), in which all physical measurements must be performed, as we are unable to observe in the rest frame $\Sigma'(x'^{\mu})$ of a micro-particle, is also the starting point of our physical assumptions. We obtain further aspects like Feynman's relativistic parton theory, if we formulate the connection between the rest system and observation space from an axiomatic point of view, where the problem of time-synchronisation for elementary particles becomes the starting point for a generalisation of the quantisation procedure (4.1). The Heisenberg equation does not show any connection to the parton (or quark) model. If we had neglected the inhomogeneous Lorentz group as a part of the operator equation (4.25) and taken account of the term $\Gamma_\nu^{\text{Cartan}}$ only, as in equation (5.11), we would have obtained the Heisenberg equation and therefore no relation to the parton model, as it is necessary to introduce the Poincaré group between measurement and rest frame as a term of an operator equation. By that an elementary particle is no longer described as a 'point'. However, the neglect of the term Γ_ν is only justified by mathematical simplification and the exact solution (5.1).

If the point of view is justified that Riemannian geometry (or $\Gamma_\nu^{\text{Riemann}} = 0$) is not essential in the theory of elementary particles, and a space-time structure with *constant but discrete* curvature and torsion ($\Gamma_\nu^{\text{Cartan}} \neq 0$) is sufficient, then we are able to consider the Heisenberg term and the parton model as different aspects in the theory of elementary particles. It is to be hoped that a combination of a quantised de Sitter space, leading to a generalisation of the parton model and the torsion of space will lead to a suitable generalisation of the Heisenberg equation. By the stipulation that the affine connection represents only torsion, where $\{{}^{\alpha}_{\nu\mu}\} \equiv 0$, we obtain from equations (4.24) and (2.6b):

$$\begin{aligned} \gamma_\mu (L_\nu{}^\mu x^\nu + a^\mu)\phi &= -l^2 \gamma^\nu [\phi,_\nu \pm \Gamma_\nu^{\text{Cartan}} \phi] \\ \gamma^\nu \Gamma_\nu^{\text{Cartan}} &= l^2 \gamma^\nu \gamma (\bar{\phi} \gamma_\nu \gamma \phi) \phi \end{aligned} \qquad (5.12)$$

This field equation, representing a combination of torsion of space and the parton model, is more difficult and complicated than equations (2.1) and (4.25). The author could not find nontrivial solutions. However, the nonlinear term of equation (5.12) may play the role of a perturbation. But it is significant for the calculation of transition probabilities. As it is necessary to find solutions and concrete results, we wish to return again to Feynman's parton theory. For the interaction of three partons (or quarks) Feynman introduced the operator K (equation (3.2) in Section 3) and assumed the propagator to be K^{-1}. Perturbations $-\delta K$ of this operator will then lead to a propagator

$$\hat{K}^{-1} = (K - \delta K)^{-1}$$

For the calculation of matrix elements Feynman used the relation

$$\tilde{K}^{-1} = K^{-1} + K^{-1}\delta K . K^{-1} + K^{-1}\delta K . K^{-1}\delta K . K^{-1} + \ldots +$$

By that Feynman and co-workers could consider small interactions, for example, spin-spin interactions! (This kind of interaction is described by the non-linear term of (2.1) a priori.)

In the opinion of the author it is necessary to investigate such approximation methods, and the solution (5.1) should be a suitable type of function for propagator methods. He hopes further that this paper may contribute to new formulations of questions in high energy physics. The starting point of this paper for deriving the parton model is the problem of the time-synchronisation for microparticles and the uncertainty principle. This unexpected connection between parton model and measurement process of a relativistic problem may also contribute to detailed discussions in the theory of measurement processes.

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